XI. *A Mathematical Theory of Magnetism.—Continuation of Part* I. *By* William Thomson, *Esq., M.A., F.R.S.E., Fellow of St. Peter's College, Cambridge, and Professor of Natural Philosophy in the University of Glasgow. Communicated by Lieut.-Colonel* Sabine, *For. Sec. R.S.*

Received June 20,—Read June 20, 1850.

Chapter V. *On Solenoidal and Lamellar Distributions of Magnetism.*

65. In the course of some researches upon inverse problems regarding distributions of magnetism, and upon the comparison of electro-magnets and common magnets, I have found it extremely convenient to make use of definite terms to express certain distributions of magnetism and forms of magnetized matter possessing remarkable properties. The use of such terms will be of still greater consequence in describing the results of these researches, and therefore, before proceeding to do so, I shall give definitions of the terms which I have adopted, and explain briefly the principal properties of the magnetic distributions to which they are applied. The remainder of this chapter will be devoted to three new methods of analysing the expressions for the resultant force of a magnet at any point, suggested by the consideration of these special forms of magnetic distribution. A Mathematical Theory of Electro-Magnets, and Inverse Problems regarding magnetic distributions, are the subjects of papers which I hope to be able to lay before the Royal Society on a subsequent occasion.

66. *Definitions and explanations regarding Magnetic Solenoids.*

(1.) A magnetic solenoid* is an infinitely thin bar of any form, longitudinally magnetized with an intensity varying inversely as the area of the normal section in different parts.

The constant product of the intensity of magnetization into the area of the normal section, is called the magnetic strength, or sometimes simply the strength of the solenoid. Hence the magnetic moment of any straight portion, or of an infinitely small portion of a curved solenoid, is equal to the product of the magnetic strength into the length of the portion.

(2.) A number of magnetic solenoids of different lengths may be put together so

* This term (from σωλὴν, *a tube*,) is suggested by the term "electro-dynamic solenoid" applied by Ampère to a certain tube-like arrangement of galvanic circuits which produces precisely the same external magnetic effect as is produced by ordinary magnetism distributed in the manner defined in the text. The especial appropriateness of the term to the magnetic distribution is manifest from the relation indicated in the foot-note on § 76 below, between the intensity and direction of magnetization in a solenoid, and the velocity and direction of motion of a liquid flowing through a tube of constant or varying section.

as to constitute what is, as far as regards magnetic action, equivalent to a single infinitely thin bar of any form, longitudinally magnetized with an intensity varying arbitrarily from one end of the bar to the other. Hence such a magnet may be called a complex magnetic solenoid.

The magnetic strength of a complex solenoid is not uniform, but varies from one part to another.

(3.) An infinitely thin closed ring, magnetized in the manner described in (1.), is called a closed magnetic solenoid.

67. *Definitions and explanations regarding Magnetic Shells.*

(1.) A magnetic shell is an infinitely thin sheet of any form, normally magnetized with an intensity varying inversely as the thickness in different parts.

The constant product of the intensity of magnetization into the thickness is called the magnetic strength, or sometimes simply the strength of the shell. Hence the magnetic moment of any plane portion, or of an infinitely small portion of a curved magnetic shell, is equal to the product of the magnetic strength, into the area of the portion.

(2.) A number of magnetic shells of different areas may be put together so as to constitute what is, as far as regards magnetic action, equivalent to a single infinitely thin sheet of any form, normally magnetized with an intensity varying arbitrarily over the whole sheet. Hence such a magnet may be called a complex magnetic shell.

The magnetic strength of a complex shell is not uniform, but varies from one part to another.

(3.) An infinitely thin sheet, of which the two sides are closed surfaces, is called a closed magnetic shell.

68. *Solenoidal and Lamellar Distributions of Magnetism.*—If a finite magnet of any form be capable of division into an infinite number of solenoids which are either closed or have their ends in the bounding surface, the distribution of magnetism in it is said to be solenoidal, and the substance is said to be solenoidally magnetized.

If a finite magnet of any form be capable of division into an infinite number of magnetic shells which are either closed or have their edges in the bounding surface, the distribution of magnetism in it is said to be lamellar*, and the substance is said to be lamellarly magnetized.

69. *Complex Lamellar Distributions of Magnetism.*—If a finite magnet of any form be capable of division into an infinite number of complex magnetic shells, it is said to possess a complex lamellar distribution of magnetism.

70. *Complex Solenoidal Distributions of Magnetism.*—Since, by cutting it along

* The term *lamellar*, adopted for want of a better, is preferred to "laminated"; since this might be objected to as rather meaning "composed of plates," than *composed of shells*, whether plane or curved, and is besides too much associated with a mechanical structure such as that of slate or mica, to be a convenient term for the magnetic distributions defined in the text.

lines of magnetization, every magnet of finite dimensions may be divided into an infinite number of longitudinally magnetized infinitely thin bars or rings, any distribution of magnetism which is not solenoidal might be called a complex solenoidal distribution; but no advantage is obtained by the use of this expression, which is only alluded to here, on account of the analogy with the subject of the preceding definition.

71. PROP.—*The action of a magnetic solenoid is the same as if a quantity of positive or northern imaginary magnetic matter numerically equal to its magnetic strength, were placed at one end, and an equal absolute quantity of negative or southern matter at the other end.*

The truth of this proposition follows at once from the investigation of Chap. III. §§ 36, 37, 38.

Cor. 1.—The action of a magnetic solenoid is independent of its form, and depends solely on its strength and the positions of its extremities.

Cor. 2.—A closed solenoid exerts no action on any other magnet.

Cor. 3.—The "resultant force" (defined in Chap. IV. § 49) at any point in the substance of a closed magnetic solenoid vanishes.

72. PROP.—*If* i *be the intensity of magnetization, and* ω *the area of the normal section at any point* P, *at a distance* s *from one extremity of a complex solenoid, and if* $[i\omega]$ *and* $\{i\omega\}$ *denote the values of the product of these quantities at the extremity from which* s *is measured, and at the other extremity respectively; the magnetic action will be the same as if there were a distribution of imaginary magnetic matter, through the length of the bar of which the quantity in an infinitely small portion* ds, *of the length at the point* P, *would be* $-\frac{d(i\omega)}{ds}ds$, *and accumulations of quantities equal to* $-[i\omega]$ *and* $\{i\omega\}$ *respectively at the two extremities.*

The truth of this proposition follows immediately from the conclusions of Chap. III. § 38.

73. PROP.—*The potential of a magnetic shell at any point is equal to the solid angle which it subtends at that point multiplied by its magnetic strength*.*

Let dS denote the area of an infinitely small element of the shell, Δ the distance of this element from the point P, at which the potential is considered, and θ the angle between this line, and a normal to the shell drawn through the north polar side of dS. Then if λ denote the magnetic strength of the shell, the magnetic moment of the element dS will be $\lambda\ dS$, and (§ 54.) the potential due to it at P will be

$$\frac{\lambda dS \cdot \cos\theta}{\Delta^2}.$$

* This theorem is due to GAUSS (see his paper "On the General Theory of Terrestrial Magnetism," § 38; of which a translation is published in TAYLOR's Scientific Memoirs, vol. ii.). AMPÈRE's well-known theorem, referred to by GAUSS, that a closed galvanic circuit produces the same magnetic effect as a magnetic shell of any form having the circuit for its edge, implies obviously the truth of the first part of Cor. 2 below.

Now $\frac{dS \cdot \cos\theta}{\Delta^2}$ is the solid angle subtended at P by the element dS, and therefore the potential due to any infinitely small element is equal to the product of its magnetic strength, into the solid angle which its area subtends at P. But the potential due to the whole is equal to the sum of the potentials due to the parts, and the strength is the same for all the parts. Hence the potential due to the whole shell is equal to the product of its strength into the sum of the solid angles which all its parts, or the solid angle which the whole, subtends at P.

Cor. 1.—The expression $\frac{dS \cdot \cos\theta}{\Delta^2}$, which occurred in the preceding demonstration, being positive or negative according as θ is acute or obtuse, it appears that the solid angles subtended by different parts of the shell at P must be considered as positive or negative according as their north polar or their south polar sides are towards this point.

Cor. 2.—The potential at any point due to a magnetic shell is independent of the form of the shell itself, and depends solely on its bounding line or edge, subject to an ambiguity, the nature of which is made clear by the following statement:—

If two shells of equal magnetic strength, λ, have a common boundary, and if the north polar side of one, and the south polar side of the other be towards the enclosed space, the potentials due to them at any external point will be equal; and the potential at any point in the enclosed space, due to that one of which the northern polarity is on the inside, will exceed the potential due to the other by the constant $4\pi\lambda$.

Cor. 3.—Of two points infinitely near one another on the two sides of a magnetic shell, but not infinitely near its edge, the potential at that one which is on the north polar side exceeds the potential at the other by the constant $4\pi\lambda$.

Cor. 4.—The potential of a closed magnetic shell of strength λ, with its northern polarity on the inside, is $4\pi\lambda$, for all points in the enclosed space, and 0 for all external points; and for points in the magnetized substance it varies continuously from the inside, where it is $4\pi\lambda$ to the outside, where it is 0.

Cor. 5.—A closed magnetic shell exerts no force on any other magnet.

Cor. 6.—The "resultant force" (§ 49.) at any point in the substance of a closed magnetic shell is equal to $\frac{4\pi\lambda}{\tau}$, if τ be the thickness, or to $4\pi i$, if i be the intensity of magnetization of the shell in the neighbourhood of the point, and is in the direction of a normal drawn from the point through the south polar side of the shell.

Cor. 7.—If the intensity of magnetization of an open shell be finite, the resultant force at any external point not infinitely near the edge is infinitely small; but the force at any point in the substance not infinitely near the edge is finite, and is equal to $4\pi i$, if i be the intensity of the magnetization in the neighbourhood of the point, and is in the direction of a normal through the south polar side.

74. PROP.—*A distribution of magnetism expressed by* $\{(\alpha, \beta, \gamma)$ at $(x, y, z)\}$* *is solenoidal if, and is not solenoidal unless* $\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0$.

The condition that a given distribution of magnetism, in a substance of finite dimensions, may be solenoidal, is readily deduced from the investigations of § 42, by means of the propositions of §§ 71 and 72. For, if the distribution of magnetism be solenoidal, the imaginary magnetic matter by which the polarity of the whole magnet may be represented will be situated at the ends of the solenoids, according to § 71, and therefore (§ 68.) will be spread over the bounding surface. On the other hand, if the distribution be not solenoidal, that is, if the magnet be divisible into solenoids, of which some, if not all, are complex; there will, according to § 72, be an internal distribution of imaginary magnetic matter in the representation of the polarity of the whole magnet. Hence it follows from § 42, that if α, β, γ denote the components of the intensity of magnetization at any internal point (x, y, z), the equation

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

expresses that the distribution of magnetism is solenoidal†.

75. PROP.—*A distribution of magnetism* $\{(\alpha, \beta, \gamma)$ at $(x, y, z)\}$ *is lamellar if, and is*

* Where α, β, γ, which may be called the components, parallel to the axes of coordinates, of the magnetization at (x, y, z), denote respectively the products of the intensity into the direction cosines of the magnetization.

† The analogy between the circumstances of this expression and those of the cinematical condition expressed by "the equation of continuity" to which the motion of a homogeneous incompressible fluid is subject, is so obvious that it is scarcely necessary to point it out. When an incompressible fluid flows through a tube of variable infinitely small section, the velocity (or in reality the mean velocity) in any part is inversely proportional to the area of the section. Hence the intensity and direction of magnetization, in a solenoid, according to the definition, are subject to the same law as the mean fluid velocity in a tube with an incompressible fluid flowing through it. Again, if any finite portion of a mass of incompressible fluid in motion be at any instant divided into an infinite number of solenoids (that is, tube-like parts), by following the lines of motion the velocity in any one of these parts will at different points of it be inversely proportional to the area of its section. Hence the intensity and direction of magnetization in a solenoidal distribution of magnetism, according to the definition, are subject to the same condition as the fluid-velocity and its direction, at any point in an incompressible fluid in motion. It may be remarked, that by making an investigation on the plan of § 42. to express merely the condition that there may be no internal distribution of imaginary magnetic matter, the equation $\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0$ is obtained in a manner precisely similar to a mode of investigating the equation of continuity for an incompressible fluid, now well known, which is given in DUHAMEL's Cours de Mécanique, and in the Cambridge and Dublin Mathematical Journal, vol. ii. p. 282. The following very remarkable proposition is an immediate consequence of the proposition that "a closed solenoid exerts no action on any other magnet" (§ 71, Cor. 2 above), in virtue of the analogy here indicated.

"If a closed vessel of any internal shape, be completely filled with an incompressible fluid, the fluid set into any possible state of motion, and the vessel held at rest; and if a solid mass of steel of the same shape as the space within the vessel be magnetized at each point with an intensity proportional and in a direction corresponding to the velocity and direction of the motion at the corresponding point of the fluid at any instant; the magnet thus formed will exercise no force on any external magnet."

not lamellar unless αdx$+\beta$dy$+\gamma$dz *is the differential of a function of three independent variables.*

Let ψ be a variable which has a certain value for each of the series of surfaces by which the magnet may be divided into magnetic shells; so that, if ψ be considered as a function of x, y, z, any one of these surfaces will be represented by the equation

$$\psi(x, y, z)=\Pi \quad \ldots \ldots \ldots \quad (a);$$

and the entire series will be obtained by giving the parameter, Π, successively a series of values each greater than that which precedes it by an infinitely small amount. According to the definition of a magnetic shell (§ 67.), the lines of magnetization must cut these surfaces orthogonally; and hence, since α, β, γ denote quantities proportional to the direction cosines of the magnetization at any point, we must have

$$\frac{\alpha}{\frac{d\psi}{dx}}=\frac{\beta}{\frac{d\psi}{dy}}=\frac{\gamma}{\frac{d\psi}{dz}} \quad \ldots \ldots \ldots \quad (b).$$

Let us consider the magnetic shell between two of the consecutive surfaces corresponding to values of the parameter of which the infinitely small difference is ϖ. The thickness of this shell at any point (x, y, z) will be

$$\frac{\varpi}{\left(\frac{d\psi^2}{dx^2}+\frac{d\psi^2}{dy^2}+\frac{d\psi^2}{dz^2}\right)^{\frac{1}{2}}}.$$

Now the product of the intensity of magnetization, into the thickness of the shell, must be constant for all points of the same shell; and hence, since ϖ is constant, and since α, β, γ denote quantities such that $(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}$ is the intensity of magnetization at any point, we must have

$$\frac{(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}}{\left(\frac{d\psi^2}{dx^2}+\frac{d\psi^2}{dy^2}+\frac{d\psi^2}{dz^2}\right)^{\frac{1}{2}}}=\mathrm{F}(\psi) \quad \ldots \ldots \ldots \quad (c),$$

where $\mathrm{F}(\psi)$ denotes a quantity which is constant when ψ is constant. This equation, and the two equations (b), express all the conditions required to make the given distribution lamellar. By combining them we obtain the following three, which are equivalent to them :—

$$\alpha=\mathrm{F}(\psi)\frac{d\psi}{dx}, \quad \beta=\mathrm{F}(\psi)\frac{d\psi}{dy}, \quad \gamma=\mathrm{F}(\psi)\frac{d\psi}{dz};$$

and hence, if $\int \mathrm{F}(\psi)d\psi$ be denoted by φ, we have

$$\alpha=\frac{d\varphi}{dx}, \quad \beta=\frac{d\varphi}{dy}, \quad \gamma=\frac{d\varphi}{dz} \quad \ldots \ldots \ldots \quad (\mathrm{II.}),$$

where φ is some function of x, y and z. Hence the condition that a magnetic distribution (α, β, γ) may be lamellar, is simply that $\alpha dx+\beta dy+\gamma dz$ must be the differential of a function of three independent variables. The equations to express this are obtained in their simplest forms by eliminating the arbitrary function φ by differentiation; and are of course

$$\left.\begin{aligned} \frac{d\beta}{dz}-\frac{d\gamma}{dy}=0 \\ \frac{d\gamma}{dx}-\frac{d\alpha}{dz}=0 \\ \frac{d\alpha}{dy}-\frac{d\beta}{dx}=0 \end{aligned}\right\} \quad \text{(III.)}$$

Cor.—It follows from the first part of the preceding investigation that equations (*b.*) express that the distribution, if not lamellar, is complex-lamellar. By eliminating the arbitrary function ψ from those equations, (which merely express that $\alpha dx+\beta dy+\gamma dz$ is integrable by a factor,) we obtain the well-known equation

$$\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dy}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)=0 \quad \text{(IV.)},$$

as the simplest expression of the condition that α, β, γ must satisfy, in order that the distribution which they represent may be complex-lamellar; and we also conclude that if this equation be satisfied the distribution must be complex-lamellar, unless each term of the first number vanishes by equations (III.) being satisfied, in which case the distribution is, as we have seen, lamellar.

76. The resultant force at any point external to a lamellarly-magnetized magnet will, according to § 73 (Cors. 2 and 4.), depend solely upon the edges of the shells into which it may be divided by surfaces perpendicular to the lines of magnetization (or the bands into which those surfaces cut the bounding surface), and not at all on the forms of these shells, within the bounding surface, nor upon any closed shells of which part of the magnet may consist; and the resultant force at any internal point may (§ 73. Cors. 2, 4, and 7,) be obtained by compounding a force depending solely on those edges, with a force in the direction contrary to that of the magnetization of the substance at the point, and equal to the product of 4π into the intensity of the magnetization. For either an external or an internal point, the resultant force may be expressed by means of a potential, according to § 49; and the value of this potential may be obtained by means of the theorems of § 73, in the following manner.

Let us suppose all the open shells, that is to say all the shells cut by the bounding surface of the given magnet, to be removed, and an imaginary series of shells having the same edges, and the same magnetic strengths, and coinciding with the bounding surface, substituted for them; and, for the sake of definiteness, let us suppose each of these shells to have its north polar side outwards, and to occupy a part of the surface for which the value of φ is greater than at its edge. The whole surface will thus be occupied by a series of superimposed magnetic shells, constituting a complex magnetic shell which will produce a potential at any external point the same as that due to the whole of the given magnet; and it will produce a potential at any internal point, which, together with the potential due to the closed shells which surround it, if there are any, and (§ 73, Cor. 2.) together with the product of 4π into the sum of the strengths of any open shells which have it between them and their superficial substitutes, will be the potential due to the whole of the given magnet at this point.

Now if $d\phi$ denote the difference between the values of ϕ at two consecutive surfaces of the series, by which we may conceive the whole magnet to be divided into shells, it follows, from the investigation of § 75, that the magnetic strength of the shell is equal to $d\phi$. Hence, if A denote the least value of ϕ at any part of the bounding surface, and ϕ be supposed to correspond to a point in the surface, the strength of the complex magnetic shell, found by adding the strengths of all the shells of the imaginary series superimposed at this point, will be $\phi-A$; and if P be an internal point, and the value of ϕ at it be denoted by (ϕ), the sum of the strengths of all the shells between that which passes through P and that which corresponds to A, will be $(\phi)-A$, from which it may be demonstrated*, that, whether (ϕ) be $>$ or $<A$, and whatever be the nature of the shells, whether all open or some open and some closed, the quantity to be added to the potential due to the imaginary complex shell coinciding with the surface of the magnet to find the actual potential at P, is $4\pi\{(\phi)-A\}$. Now, from what we have seen above, it follows that the potential at any point P, due to an element, dS, of this complex shell is $\frac{\phi \,.\, \cos\theta dS}{\Delta^2}$, if θ denote the angle which an external normal, or a normal through the north polar side of dS, makes with a line drawn from dS to P, and Δ the length of this line. Hence the total potential at P, due to the whole complex shell, is equal to

$$\iint \frac{\{\phi+A\}\cos\theta dS}{\Delta^2},$$

in which the integration includes the whole bounding surface of the magnet. Hence, if V denote the potential at P, we have the following expression, according as P is external or internal,—

$$V=\iint \frac{\{\phi-A\}\cos\theta dS}{\Delta^2},$$

or

$$V=\iint \frac{\{\phi-A\}\cos\theta dS}{\Delta^2}+4\pi\{(\phi)-A\}.$$

These expressions may be simplified if we remark that, for any external point,

$$\iint \frac{\cos\theta dS}{\Delta^2}=0,$$

and that, for any internal point,

$$\iint \frac{\cos\theta dS}{\Delta^2}=-4\pi$$

(since θ is the angle between the line Δ and the *external* normal through dS). We

* See second foot-note on § 48 above, and Cors. 2, 3, § 76, below.

thus obtain, for an external point,

$$V=\iint\frac{\phi\,.\cos\theta dS}{\Delta^2};$$

and for an internal point, ... (V.)

$$V=\iint\frac{\phi\,.\cos\theta dS}{\Delta^2}+4\pi(\phi),$$

Cor. 1.—The potentials at two points infinitely near one another, even if one be in the magnetized substance and the other be external, differ infinitely little; for the value of

$$\iint\frac{\phi\,.\cos\theta dS}{\Delta^2},$$

at a point infinitely near the surface and within it, is found by adding $-4\pi(\phi)$ to the value of the same expression at an external point infinitely near the former.

Cor. 2.—If the value of

$$\iint\frac{\phi\,.\cos\theta dS}{\Delta^2}$$

be denoted by $-Q$ for any internal point, x, y, z; and if $(\alpha), (\beta), (\gamma)$ denote the components of the intensity of magnetization, and X, Y, Z the components of the resultant magnetic force at this point, (that is, according to the definition in the second footnote on § 48, the force at a point in an infinitely small crevass tangential to the lines of magnetization at x, y, z) we have

$$\left.\begin{aligned} X&=-\frac{dV}{dx}=\frac{dQ}{dx}-4\pi(\alpha)\\ Y&=-\frac{dV}{dy}=\frac{dQ}{dy}-4\pi(\beta)\\ Z&=-\frac{dV}{dz}=\frac{dQ}{dz}-4\pi(\gamma)\end{aligned}\right\}\quad\ldots\ldots\ldots\quad \text{(VI.)}$$

The resultant of the partial components, $-4\pi(\alpha), -4\pi(\beta), -4\pi(\gamma)$, is a force equal to $4\pi(i)$ acting in a direction contrary to that of magnetization, and this, compounded with the resultant of

$$\frac{dQ}{dx},\ \frac{dQ}{dy},\ \frac{dQ}{dz},$$

which depends solely on the edges of the shells, gives the total resultant force at the internal point. We thus see precisely how the statements made at the commencement of § 76. are fulfilled.

Cor. 3. It is obvious, by the preceding investigation, that

$$\frac{dQ}{dx},\ \frac{dQ}{dy},\ \frac{dQ}{dz}$$

are the components of the force at a point in an infinitely small crevass perpendicular to the lines of magnetization at x, y, z.

77. An analytical demonstration of these expressions may be obtained by a partial

integration of the general expression for the potential in the case of a lamellar distribution, in the following manner:—

In equation (5) of § 55, which, as was remarked in the foot-note, expresses the potential for any point, whether internal or external, let $\frac{d\varphi}{dx}$, $\frac{d\varphi}{dy}$, and $\frac{d\varphi}{dz}$ be substituted in place of *il*, *im*, and *in* respectively; and, for the sake of brevity, let

$$\{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2\}^{\frac{1}{2}}$$

be denoted by Δ: then observing that $\frac{\xi-x}{\Delta^3}=\frac{d\frac{1}{\Delta}}{dx}$, and so for the similar terms; we have

$$\mathrm{V}=\iiint\left(\frac{d\varphi}{dx}\frac{d\frac{1}{\Delta}}{dx}+\frac{d\varphi}{dy}\frac{d\frac{1}{\Delta}}{dy}+\frac{d\varphi}{dz}\frac{d\frac{1}{\Delta}}{dz}\right)dxdydz. \quad \ldots \ldots \quad (a)$$

Dividing the second member into three terms, integrating the first by parts commencing with the factor $\frac{d\varphi}{dx}\,dx$, and so for the other terms; we obtain

$$\mathrm{V}=\left[\iint\varphi\left(\frac{d\frac{1}{\Delta}}{dx}dydz+\frac{d\frac{1}{\Delta}}{dy}dzdx+\frac{d\frac{1}{\Delta}}{dz}dxdy\right)\right]-\iiint\varphi\left(\frac{d^2\frac{1}{\Delta}}{dx^2}+\frac{d^2\frac{1}{\Delta}}{dy^2}+\frac{d^2\frac{1}{\Delta}}{dz^2}\right)dxdydz, \quad \ldots \quad (b)$$

where the brackets which inclose the double integral denote that it has reference to the surface of the body. Now, for any set of values of x, y, z, for which $\frac{1}{\Delta}$ is finite, we have, as is well known,

$$\frac{d^2\frac{1}{\Delta}}{dx^2}+\frac{d^2\frac{1}{\Delta}}{dy^2}+\frac{d^2\frac{1}{\Delta}}{dx^2}=0; \quad \ldots \ldots \ldots \quad (c)$$

and consequently, if the point ξ, η, ζ is not in the space included by the triple integral in the expression for V, each element of this integral, and therefore also the whole, vanishes. In the contrary case, the simultaneous values $x=\xi$, $y=\eta$, and $z=\zeta$ will be included in the limits of integration, and, as these values make $\frac{1}{\Delta}$ infinitely great, the equation (c) will fail for one element of the integral, although it still holds for all elements corresponding to points at a finite distance from (ξ, η, ζ). Hence, if (φ) denote the value assumed by the function φ at this point, we have

$$\iiint\varphi\left(\frac{d^2\frac{1}{\Delta}}{dx^2}+\frac{d^2\frac{1}{\Delta}}{dy^2}+\frac{d^2\frac{1}{\Delta}}{dz^2}\right)dxdydz=(\varphi)\iiint\left(\frac{d^2\frac{1}{\Delta}}{dx^2}+\frac{d^2\frac{1}{\Delta}}{dy^2}+\frac{d^2\frac{1}{\Delta}}{dx^2}\right)dxdydz,$$

where the limits of integration may correspond to any surface whatever which completely surrounds the point (ξ, η, ζ). Now it is easily proved (as is well known) that the value of

$$\iiint\left(\frac{d^2\frac{1}{\Delta}}{dx^2}+\frac{d^2\frac{1}{\Delta}}{dy^2}+\frac{d^2\frac{1}{\Delta}}{dz^2}\right)dxdydz$$

is -4π, when (ξ, η, ζ) is included in the limits of integration; and therefore the value of the triple integral, in the expression for V, is $-4\pi(\phi)$. Hence, according as the point (ξ, η, ζ) is external or internal with reference to the magnet, the potential at it is given by the expressions

$$\left.\begin{array}{ll} (1.) & V=\left[\iint\phi.\left(\frac{d\frac{1}{\Delta}}{dx}dydz+\frac{d\frac{1}{\Delta}}{dy}dzdx+\frac{d\frac{1}{\Delta}}{dz}dxdy\right)\right] \\ \text{or} \quad (2.)^* & V=\left[\iint\phi.\left(\frac{d\frac{1}{\Delta}}{dx}dydz+\frac{d\frac{1}{\Delta}}{dy}dzdx+\frac{d\frac{1}{\Delta}}{dz}dxdy\right)\right]+4\pi(\phi) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (VII.)$$

These agree with the expressions obtained above in § 76; the same double integral with reference to the surface being here expressed symmetrically by means of rectangular coordinates.

78. The value of ϕ at any point in the surface of the magnet, which, as appears from the preceding investigations, is all that is necessary for determining the potential due to a lamellar magnet at any point not contained in the magnetized substance, may, according to well-known principles, be determined by integration, if the tangential component of the magnetization at every point of the magnet infinitely near its surface be given. It appears therefore that, if it be known that a magnet is lamellarly magnetized throughout its interior, it is sufficient to have given the tangential component of its magnetization at every point infinitely near the surface or to have enough of data for determining it, without any further specification regarding the interior distribution than that it is lamellar, to enable us to determine completely its external magnetic action. This conclusion is analogous to a conclusion which may be drawn, for the case of a solenoidal distribution, from the expression obtained in § 51, for the potential of a magnet of any kind. For, from this expression, we have, according to § 74, the following in the case of a solenoidal distribution:—

$$V=\left[\iint\frac{(l\alpha+m\beta+n\gamma)dS}{\Delta}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (VIII.);$$

from which we conclude, that without farther data regarding the interior distribution than that it is solenoidal, it is sufficient to have given the normal component of the magnetization at every point infinitely near the surface to enable us to determine the external magnetic action. Yet, although analogous conclusions are thus drawn from these two formulæ, the formulæ themselves are not analogous, as the former (that of § 51) is applicable to all distributions, whether solenoidal or not, and shows precisely how the resultant magnetic action will in general depend on the interior distribution besides the normal magnetization near the surface, according to the deviation from

* It may be proved that the force derived from a potential having the same expression (VII.) (1.) as for external points, is, for any internal point, the force at a point within an infinitely small crevass perpendicular to the lines of magnetization; as it is easily shown that the differential coefficients of $4\pi(\phi)$ are the rectangular components of the force at such a point due to the free contrary polarities on the two sides of the crevass.

being solenoidal which it presents; while the formulæ of § 76. merely expresses a fact with reference to lamellar distributions, and being only applicable to lamellar distributions, do not indicate the effect of a deviation from being lamellar, in a distribution of a general form. Certain considerations regarding the comparison between common magnets and electro-magnets, suggested by AMPÈRE's theorem that the magnetic action of a closed galvanic circuit is the same as that of a "magnetic shell" (as defined in § 67.) of any form having its edge coincident with the circuit, led me to a synthetical investigation of a distribution of galvanism through the interior and at the surface of a magnet magnetized in any arbitrary manner, from which I deduced formulæ, for the resultant force at any external or internal point, giving the desired indication regarding effect of a deviation from being lamellar, on expressions which, for lamellar distributions, depend solely on the tangential component of magnetization at points infinitely near the surface. These galvanic elements throughout the body, from the action of which the resultant force at any external point is compounded, produce effects which are not separately expressible by means of a potential, and therefore, although of course when the three components X, Y, Z of the total resultant force at any point (x, y, z) have been obtained, they will be found to be such that $Xdx+Ydy+Zdz$ is a complete differential, the separate infinitely small elements of which these forces are compounded by integration with reference to the elements of the magnet, do not separately satisfy such a condition. Hence the investigation does not lead to an expression for the potential; but by means of it the following expressions for the three components of the force at any external point, or on a point within any infinitely small crevass perpendicular to the lines of magnetization, have been obtained*:—

$$\left.\begin{aligned}
X&=\iiint dxdydz\left\{\frac{\eta-y}{\Delta^3}\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)-\frac{\zeta-z}{\Delta^3}\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)\right\}-\left[\iint\left\{\frac{\eta-y}{\Delta^3}(m\alpha-l\beta)-\frac{\zeta-z}{\Delta^3}(l\gamma-n\alpha)\right\}dS\right]\\
Y&=\iiint dxdydz\left\{\frac{\zeta-z}{\Delta^3}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)-\frac{\xi-x}{\Delta^3}\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)\right\}-\left[\iint\left\{\frac{\zeta-z}{\Delta^3}(n\beta-m\gamma)-\frac{\xi-x}{\Delta^3}(m\alpha-l\beta)\right\}dS\right]\\
Z&=\iiint dxdydz\left\{\frac{\xi-x}{\Delta^3}\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)-\frac{\eta-y}{\Delta^3}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)\right\}-\left[\iint\left\{\frac{\xi-x}{\Delta^3}(l\gamma-n\alpha)-\frac{\eta-y}{\Delta^3}(n\beta-m\gamma)\right\}dS\right]
\end{aligned}\right\}\text{(IX.)}$$

The investigation by which I originally obtained these expressions is, with reference to galvanism, precisely analogous to the investigation in § 42. with reference to imaginary magnetic matter. It cannot be given without explanations regarding the elements of electro-magnetism which would exceed the limits of the present communication; but when I had once discovered the formulæ I had no difficulty in working out the subjoined analytical demonstration of them for the case of an external point,

* The expression $Xdx+Ydy+Zdz$ will not be a complete differential for internal points, unless the distribution of magnetism be lamellar, since, for any internal point, X, Y, Z differ from the rectangular components of the resultant force, as defined in § 48, by the quantities $4\pi\alpha$, $4\pi\beta$, $4\pi\gamma$, respectively, and since (§ 52) the resultant force, for all points, whether internal or external, is derivable from a potential.

which is precisely analogous to POISSON's original investigation (shown in § 56. of this paper) of the formula of § 51.

79. Equations (3) and (4) of §§ 51. and 52, lead to expressions for the components of the resultant force at any point in the neighbourhood of a magnet. Taking only one of them, (since the three expressions are symmetrical) that for X for instance, we have

$$X=-\frac{d}{d\xi}\iiint dxdydz\left\{\alpha\frac{d\frac{1}{\Delta}}{dx}+\beta\frac{d\frac{1}{\Delta}}{dy}+\gamma\frac{d\frac{1}{\Delta}}{dz}\right\}.$$

Now if the factor of $dxdydz$ in the second member of this equation be differentiated with reference to ξ, an expression is obtained which does not become infinitely great for any values of x, y, z included within the limits of integration, since the point (ξ, η, ζ) is considered to be external in the present investigation. Hence the differentiation with reference to ξ may be performed under the integral sign; and, since

$$\frac{d\frac{1}{\Delta}}{d\xi}=-\frac{d\frac{1}{\Delta}}{dx},$$

we thus obtain

$$X=\iiint dxdydz\left\{\alpha\frac{d^2\frac{1}{\Delta}}{dx^2}+\beta\frac{d^2\frac{1}{\Delta}}{dy^2}+\gamma\frac{d^2\frac{1}{\Delta}}{dz^2}\right\}^*.$$

Now, for all points included within the limits of integration, we have, from LAPLACE's well-known equation,

$$\frac{d^2\frac{1}{\Delta}}{dx^2}=-\left(\frac{d^2\frac{1}{\Delta}}{dy^2}+\frac{d^2\frac{1}{\Delta}}{dz^2}\right),$$

and therefore

$$X=\iiint dxdydz\left\{-\alpha\left(\frac{d^2\frac{1}{\Delta}}{dy^2}+\frac{d^2\frac{1}{\Delta}}{dz^2}\right)+\beta\frac{d^2\frac{1}{\Delta}}{dxdy}+\gamma\frac{d^2\frac{1}{\Delta}}{dxdz}\right\}.$$

* If the point (ξ, η, ζ) be either within the magnet, or infinitely near it, the factor of $dxdydz$ in this integral is infinitely great for values of (x, y, z) included within the limits of integration; and it may be demonstrated that the value of a part of the integral corresponding to any infinitely small portion of the magnet infinitely near the point (ξ, η, ζ) is in general finite, and that it depends on the form of this portion, on its position with reference to the line of magnetization through (ξ, η, ζ), and on the proportions of the distances of its different parts from this point. It follows that if the point ξ, η, ζ be internal, and if a portion of the magnet round it be omitted from the integral, the value of the integral will be affected by the form of the omitted portion, however small its dimensions may be, and consequently the complete integral has no determinate value if the point (ξ, η, ζ) be internal. Hence, although as we have seen above (§§ 51, 51.),

$$-\frac{d}{d\xi}\iiint dxdydz\left\{\alpha\frac{d\frac{1}{\Delta}}{dx}+\beta\frac{d\frac{1}{\Delta}}{dy}+\gamma\frac{d\frac{1}{\Delta}}{dz}\right\}$$

has in all cases a determinate value, which, by the definition (§ 48.), is called the component parallel to OX of the resultant force at (ξ, η, ζ), the expression

$$-\iiint dxdydz\frac{d}{d\xi}\left\{\alpha\frac{d\frac{1}{\Delta}}{dx}+\beta\frac{d\frac{1}{\Delta}}{dy}+\gamma\frac{d\frac{1}{\Delta}}{dz}\right\}$$

has no meaning when (ξ, η, ζ) is in the substance of the magnet.

Dividing the second member into four terms, and applying an obvious process of integration by parts, we deduce

$$X=\left[\iint\left\{-\alpha\frac{d\frac{1}{\Delta}}{dy}dxdz-\alpha\frac{d\frac{1}{\Delta}}{dz}dxdy+\beta\frac{d\frac{1}{\Delta}}{dy}dydz+\gamma\frac{d\frac{1}{\Delta}}{dz}dydz\right\}\right.$$
$$\left.+\iiint dxdydz\left\{\frac{d\alpha}{dy}\frac{d\frac{1}{\Delta}}{dy}+\frac{d\alpha}{dz}\frac{d\frac{1}{\Delta}}{dz}-\frac{d\beta}{dx}\frac{d\frac{1}{\Delta}}{dy}-\frac{d\gamma}{dz}\frac{d\frac{1}{\Delta}}{dz}\right\}\right].$$

Modifying the double integral by assuming, in its different terms,

$$dydz=ld\text{S};\quad dzdx=md\text{S};\quad dxdy=nd\text{S},$$

and altering the order of all the terms, we obtain

$$X=\iiint dxdydz\left\{\frac{d\frac{1}{\Delta}}{dy}\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)-\frac{d\frac{1}{\Delta}}{dz}\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)\right\}-\left[\iint\left\{\frac{d\frac{1}{\Delta}}{dy}(m\alpha-l\beta)-\frac{d\frac{1}{\Delta}}{dz}(l\gamma-n\alpha)\right\}d\text{S}\right].$$

This expression, when the indicated differentiations are actually performed upon $\frac{1}{\Delta}$, becomes identical with the expression for X at the end of § 78, and the formulæ which it was required to prove are therefore established.

80. The triple integrals in these expressions vanish in the case of a lamellar distribution, in virtue of the equations (III.) of §§ 75; and we have simply

$$\left.\begin{aligned}X&=-\left[\iint\left\{\frac{d\frac{1}{\Delta}}{dy}(m\alpha-l\beta)-\frac{d\frac{1}{\Delta}}{dz}(l\gamma-n\alpha)\right\}d\text{S}\right]\\Y&=-\left[\iint\left\{\frac{d\frac{1}{\Delta}}{dz}(n\beta-m\gamma)-\frac{d\frac{1}{\Delta}}{dx}(m\alpha-l\beta)\right\}d\text{S}\right]\\Z&=-\left[\iint\left\{\frac{d\frac{1}{\Delta}}{dx}(l\gamma-n\alpha)-\frac{d\frac{1}{\Delta}}{dy}(n\beta-m\gamma)\right\}d\text{S}\right]\end{aligned}\right\}\quad\ldots\ldots\quad\text{(X.)}.$$

To interpret these expressions, let us assume, for brevity,

$$U=n\beta-m\gamma;\quad V=l\gamma-n\alpha;\quad W=m\alpha-l\beta\quad\ldots\ldots\ldots\quad\text{(XI.)}.$$

From these we deduce

$$\left.\begin{aligned}mW-nV&=\alpha-l\,(l\alpha+m\beta+n\gamma)=\alpha_{\prime}\\nU-lW&=\beta-m(l\alpha+m\beta+n\gamma)=\beta_{\prime}\\lV-mU&=\gamma-n\,(l\alpha+m\beta+n\gamma)=\gamma_{\prime}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad\text{(XII.)};$$

where $\alpha_{\prime}$, $\beta_{\prime}$, $\gamma_{\prime}$ denote the rectangular components of the tangential component of the magnetization at a point infinitely near the surface. Conversely, from these equations we deduce

$$U=n\beta_{\prime}-m\gamma_{\prime};\quad V=l\gamma_{\prime}-n\alpha_{\prime};\quad W=m\alpha_{\prime}-l\beta_{\prime}\quad\ldots\ldots\ldots\quad\text{(XIII.)}.$$

Now the direct data required for obtaining the values of X, Y, and Z, by means of formulæ (X.), are simply the values of U, V, W at all points of its surface. Equations (XII.) show that with these data the values of $\alpha_{\prime}$, $\beta_{\prime}$, $\gamma_{\prime}$ may be calculated; and again, equations show conversely that if $\alpha_{\prime}$, $\beta_{\prime}$, $\gamma_{\prime}$ be given the required data for the problem may be immediately deduced. We infer that the necessary and sufficient data for determining the resultant force of a lamellar magnet, at any external point, by means of formulæ (X.), are equivalent to a specification of the direction and magnitude of the tangential component of the intensity of magnetization at every point infinitely near the surface of the magnet; and we conclude, as we did in § 78 from a very different process of reasoning, that besides these data, nothing but that it is lamellar throughout need be known of the interior distribution.

81. The close analogy which exists between solenoidal and lamellar distributions of magnetism having led me to the new formulæ which have just been given, it occurred to me that a formula (or formulæ, if it were necessary here to separate the cases of internal and external points,) for solenoidal distributions analogous to the formulæ (VII.) of §§ 77 for lamellar distributions might be discovered. Taking an analytical view of the problem (the synthetical view, although itself much more obvious, not showing any very obvious way of arriving at a formula of the desired kind), I observed that the formula $\iint \frac{\varphi \cdot \cos \theta dS}{\Delta^2}$ is deduced from the general expression for the potential, by a partial integration performed upon factors involving α, β, γ, and depending on the integrability of the function $\alpha dx+\beta dy+\gamma dz$, ensured by the equations

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=0, \quad \frac{d\gamma}{dx}-\frac{d\alpha}{dz}=0, \quad \frac{d\alpha}{dy}-\frac{d\beta}{dx}=0,$$

for a lamellar distribution; and I endeavoured to find a corresponding mode of treatment for solenoidal distributions, to consist of a partial integration, commencing still with factors involving α, β, γ, but depending now upon the single equation

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a),$$

instead of three equations required in the former process. After some fruitless attempts to connect this equation with the integrability of some function of two independent variables, I fell upon the following investigation, which exactly answered my expectations.

82. In virtue of the preceding equation (a), we may assume

$$\alpha=\frac{dH}{dy}-\frac{dG}{dz}, \quad \beta=\frac{dF}{dz}-\frac{dH}{dx}, \quad \gamma=\frac{dG}{dx}-\frac{dF}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV.}),$$

where F, G, H are three functions to a certain extent arbitrary, which, as I have since found, have for their most general expressions

$$\left.\begin{aligned} F&=\iint dydz\left(\frac{d\beta}{dy}-\frac{d\gamma}{dz}\right)+\frac{d\psi}{dx} \\ G&=\iint dzdx\left(\frac{d\gamma}{dz}-\frac{d\alpha}{dx}\right)+\frac{d\psi}{dy} \\ H&=\iint dxdy\left(\frac{d\alpha}{dx}-\frac{d\beta}{dy}\right)+\frac{d\psi}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XV.})$$

where ψ denotes an absolutely arbitrary function; and the indicated integrations are indefinite, with the arbitraries which they introduce subject to the equations (XIV.).

The demonstration of these equations follows immediately from the results obtained by differentiating the three equations (XIV.) with reference to x, y and z respectively. The simplest final forms for F, G and H are the following, which are deduced from the preceding by integration:—

$$\left.\begin{array}{l} \mathrm{F}=\int(\beta dz-\gamma dy)+\frac{d\psi}{dx} \\ \mathrm{G}=\int(\gamma dx-\alpha dz)+\frac{d\psi}{dy} \\ \mathrm{H}=\int(\alpha dy-\beta dx)+\frac{d\psi}{dz} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{XVI.})$$

Making substitutions according to the formulæ (XIV.) for α, β, γ in the general expression for the potential, we have

$$\mathrm{V}=\iiint dxdydz\left\{\left(\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz}\right)\frac{d\frac{1}{\Delta}}{dx}+\left(\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{dx}\right)\frac{d\frac{1}{\Delta}}{dy}+\left(\frac{d\mathrm{G}}{dx}-\frac{d\mathrm{F}}{dy}\right)\frac{d\frac{1}{\Delta}}{dz}\right\}.$$

Dividing the second member into six terms, and integrating each by parts, commencing upon the factors such as $\frac{d\mathrm{H}}{dy}\,dy$, we obtain an expression, with a triple integral involving six terms which destroy one another two and two because of properties such as

$$\frac{d}{dy}\frac{d\frac{1}{\Delta}}{dx}=\frac{d}{dx}\frac{d\frac{1}{\Delta}}{dy};$$

and besides, a double integral, which may be reduced in the usual manner to a form involving $d\mathrm{S}$, an element of the surface. We thus obtain, finally,

$$\mathrm{V}=\left[\iint\left\{(m\mathrm{H}-n\mathrm{G})\frac{d\frac{1}{\Delta}}{dx}+(n\mathrm{F}-l\mathrm{H})\frac{d\frac{1}{\Delta}}{dy}+(l\mathrm{G}-m\mathrm{F})\frac{d\frac{1}{\Delta}}{dz}\right\}d\mathrm{S}\right] \quad . \quad . \quad . \quad (\mathrm{XVII.})$$

83. The second member of this equation expresses the potential of a certain distribution of magnetism in an infinitely thin sheet coinciding with the surface of the body; the total magnetic moment of the magnetism in the area $d\mathrm{S}$ being

$$\{(m\mathrm{H}-n\mathrm{G})^2+(n\mathrm{F}-l\mathrm{H})^2+(l\mathrm{G}-m\mathrm{F})^2\}^{\frac{1}{2}}d\mathrm{S},$$

and its direction cosines proportional to

$$m\mathrm{H}-n\mathrm{G},\quad n\mathrm{F}-l\mathrm{H},\quad l\mathrm{G}-m\mathrm{F}.$$

Now we have identically,

$$l(m\mathrm{H}-n\mathrm{G})+m(n\mathrm{F}-l\mathrm{H})+n(l\mathrm{G}-m\mathrm{F})=0;$$

and hence the direction of this imaginary magnetization at every point of the surface is perpendicular to the normal. It follows that we have found a distribution of tangential magnetism in an infinitely thin sheet coinciding with the bounding surface which produces the same potential at any point, internal or external, as the given solenoidal magnet.

84. The same general conclusion may be arrived at synthetically in a very obvious manner, by taking into account the property of a solenoid stated in § 71, according to which it appears that any two solenoids of equal strength, with the same ends, produce the same force at any point whether in the magnetized substance of either, or not. For it follows from this, that when a magnet is divisible into solenoids with their ends on its surface, by joining the two ends of each solenoid by any arbitrary curve on this surface and laying a solenoid of equal strength along this curve, we obtain a series of solenoids, constituting by their superposition, a tangential distribution of magnetism in an infinitely thin sheet coinciding with the bounding surface, which produces the same resultant force at any internal or external point as the given magnet. It is not, however, easy to deduce from this synthesis, a formula involving the requisite arbitrary functions to express a superficial distribution satisfying the existing conditions in the most general manner. The analytical investigation given above, supplies, in reality, a complete solution of this problem.

It may be remarked that the sole condition which F, G and H, considered as functions of the coordinates, x, y, z, of some point in the surface of the magnet, and therefore functions of two independent variables, must satisfy in order that (XVII.) may express correctly the potential at any point—

$$l\left(\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz}\right)+m\left(\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{dx}\right)+n\left(\frac{d\mathrm{G}}{dx}-\frac{d\mathrm{F}}{dy}\right)=l\alpha+m\beta+n\gamma, \quad . \quad . \quad (\mathrm{XVIII.}),$$

x, y and z of course being supposed to satisfy the equation to the surface; and it may be proved, by a demonstration independent of the investigation which has been given, that the second member of (XVII.) has the same value for any functions F, G, H whatever, which are subject to this relation.

END OF CHAPTER V.